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A MATHEMATICAL MODEL OF THE ILLINOIS INTERLIBRARY LOAN NETWORK: REPORT NO. 5

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OF THE

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FORWARD

This is the fifth in a series of reports resulting from a research grant to the Coordinated Science Laboratory of the University of Illinois at Urbana-Champaign. The sponsor of the grant is the Illinois State Library under the Illinois Program for Title I of the Federal Library Services and Construction Act.

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1. W. B. Rouse, J. L. Divilbiss, and S. H. Rouse, A Mathematical Model of the Illinois Interlibrary Loan Network: Project Report No. 1, Coordinated Science Laboratory Report No. T-14, University of Illinois at Urbana-Champaign, November 1974, ERIC No. ED 101 667.

   Includes a review of the literature on interlibrary loan networks, a flow chart description of the Illinois network, a review of methodologies appropriate to modeling networks, an initial model, and discussion of alternative computer and communication technologies.

2. W. B. Rouse, J. L. Divilbiss, and S. H. Rouse, A Mathematical Model of the Illinois Interlibrary Loan Network: Project Report No. 2, Coordinated Science Laboratory Report No. T-16, University of Illinois at Urbana-Champaign, March 1975, ERIC No. ED 107 287.

   Includes a derivation of the mathematical model (version no. 2) and its applications to a hypothetical network so as to illustrate various policy issues. A summary of the User's Manual for the interactive program of the model is also included.

3. W. B. Rouse and S. H. Rouse, A Mathematical Model of the Illinois Interlibrary Loan Network: Project Report No. 3, Coordinated Science Laboratory Report No. T-26, University of Illinois at Urbana-Champaign, April 1976, ERIC No. ED 124 179.

   Includes a detailed analysis of the Illinois network based on data obtained from the Illinois State Library and from the Library Research Center at the University of Illinois. Several alternative request routing policies are considered and specific recommendations are discussed.

4. W. B. Rouse and S. H. Rouse, A Mathematical Model of the Illinois Interlibrary Loan Network: Project Report No. 4, Coordinated Science Laboratory Report No. T-28, University of Illinois at Urbana-Champaign, June 1976, ERIC No. ED 126 879.

   A procedure is developed for predicting the impact of computer technology on interlibrary loan activities. Specific examples include automated circulation systems and shared cataloging networks. The Illinois network is analyzed to determine the value of location and availability information as provided by the alternative computer technologies.

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SUMMARY

This report is based on a paper prepared for presentation at the 1976 Joint National Meeting of the Operations Research Society of America and The Institute of Management Sciences. The purpose of this summary is to discuss in general the mathematical derivations presented in this report and thereby allow the reader to move immediately to the example and discussion in the later part of the report and avoid the detailed equations.

The library network model which we have developed predicts probability of satisfying a request P, average time from initiation of a request until receipt of the desired item W, and average cost to satisfy a request C. To make these predictions, the model requires input data or parameters. These parameters are formed using samples of data from the network of interest. (See discussion in Project Report No. 3.) If the size of the sample is small, then our parameter estimates will be poor and we will be uncertain of the "true" parameters. As the size of the sample increases, uncertainty decreases. Uncertainty in the model parameters will lead to uncertainty in the model's predictions.

The primary purpose of this paper is to show how model parameter uncertainty affects prediction uncertainty as a function of sample size and number of request classes. As sample size increases, uncertainty decreases. However, as the number of request classes increases, the sample size in each class decreases and thus, uncertainty increases. The method developed in this report can be used as a guide for choosing sample size and number of request classes.

An example, utilizing demand data from Project Report No. 3, shows how the modified coefficient of variation ($1.96\ \sigma_{\bar{x}}/\bar{x}$) is affected by sample size (% year) and number of request classes. For a given value of the modified coefficient of variation (say α), we can say that we are 95% confident that the true average performance is within 100α percent of the estimated average performance. Choosing a desired maximum for α defines the minimum sample size and/or maximum number of request classes.

In an appendix, a new derivation of the model is presented that requires fewer assumptions and allows more robust representations of library networks.

INTRODUCTION

In an earlier paper, the development of a library network model was reported [1]. The model predicts network performance as it is affected by network configuration in terms of operating policies and alternative technologies. Network performance is defined as having several components: probability of satisfying a request, average time from initiation of a request until receipt of the desired item, average unit and total operating costs, and average request processing loads throughout the network.

Since the model was developed in 1974-1975, it has been applied to analyzing the Illinois Library and Information Network (ILLINET) as a whole [2] assessing the impact of computer technology (e.g., OCLC) on interlibrary loan networks in general and ILLINET in particular [3], and analyzing one of the regional networks of ILLINET [4].

The purpose of this paper is twofold. First, an Appendix of this paper presents a new version of the model that is more general and allows representation of many different types of networks. Second, and perhaps more important, this paper will consider estimation of the model parameters and present a method for determining how much data should be collected and how it should be aggregated. As the reader will see, parameter estimation is quite straightforward while the question of appropriate sample sizes and levels of aggregation is more complicated.

PARAMETER ESTIMATION

There are four classes of model parameters: demands, probabilities of success, processing times, and delivery times. In this section, we will define these parameters, briefly note how they can be estimated, and point out how appropriate confidence intervals can be determined.

The average number of requests per unit time generated by requesting library k in request class j is denoted by λ_{jk}; $j = 1,2,...,J$; $k = 1,2,...,K$. A requesting library is a request-gathering organization that can deal directly with the network. Thus, requesting libraries need not be "libraries" in the traditional sense. Request class can represent subject area, type of request (e.g., document or information), type of requestor, etc.

The average demand as defined by λ_{jk} is estimated in the usual manner. The general approach to determining confidence intervals for λ_{jk} can be utilized [5, Chapter 11]. If we assume that request generation can be modeled as a Poisson process, a very simple formulation is possible [6, p. 354].

The probability that resource library i will satisfy a class j request at the ℓ^{th} stage of referral is denoted by $p_{ij|\ell}$; $j = 1,2,...,I$; $j = 1,2,...,J$; $\ell = 1,2,...,L$. This probability is estimated in the usual manner with the confidence interval being straightforward [7, pp. 216-219].

The average time for resource library i to successfully process a class j request at the ℓ^{th} stage of referral is denoted by $\hat{w}_{ij\ell}$ while the average time to unsuccessfully process such a request is denoted by $\tilde{w}_{ij\ell}$. These estimates and confidence intervals are calculated in the usual way.

The average time for a class j item to be delivered from resource library i to requesting library k is denoted by t_{ijk}. Again, the estimate and confidence interval can be found in a straightforward manner.

The above discussion of parameter and confidence interval estimation was quite terse, reflecting the fact that the appropriate procedures can be found in any of a number of undergraduate textbooks. We will now consider how uncertainty in the model parameters leads to uncertainty in the predictions of the model.

DATA COLLECTION

Our network performance measures include*

$$P = \sum_{j=1}^{J} \sum_{k=1}^{K} \lambda_{jk} p_{jk} / \Lambda \tag{1}$$

$$W = \sum_{j=1}^{J} \sum_{k=1}^{K} \lambda_{jk} p_{jk} w_{jk} / P\Lambda \tag{2}$$

$$C = \sum_{j=1}^{J} \sum_{k=1}^{K} \lambda_{jk} p_{jk} c_{jk} / P\Lambda \tag{3}$$

where

$$\Lambda = \sum_{j=1}^{J} \sum_{k=1}^{K} \lambda_{jk} \tag{4}$$

and P is the probability of the network satisfying a request, W is the average time from initiation of a request until the desired item is received by the requesting library, C is the average cost per satisfied request, and Λ is the average total demand on the network.

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*The equations utilized throughout this section represent the manipulation of parameter estimates as opposed to the "true" parameter values which one usually cannot obtain. To simplify notation, we have not annotated our symbols in any way to show that they are estimates.

The components of equations 1 through 4 are defined in the Appendix in terms of the model parameters discussed in the previous section of this paper. We now want to consider how uncertainty in the model parameters results in uncertainty in P, W, and C. It will be assumed that the estimated parameters are random variables drawn from appropriate sampling distributions. Further, the quotients of the sample variances and sample sizes will be utilized as estimates of the variances of the sampling distributions.

Observation of equations 1 through 4 and those in the Appendix shows that we must frequently consider sums, products, and quotients of random variables. Thus, let us review a little probability theory before we begin our analysis.

If $y = x_1 + x_2 + \ldots + x_N$ where $x_1, x_2, \ldots, x_N$ are random variables, then

$$E(y) = \sum_{i=1}^{N} E(x_i) \tag{5}$$

and if $x_1, x_2, \ldots, x_N$ are linearly independent, then [7, p. 108]

$$\sigma_y^2 = \sum_{i=1}^{N} \sigma_{x_i}^2 \tag{6}$$

where $E(\cdot)$ denotes the expected value while σ^2 denotes the variance.

If $y = x_1 x_2 \ldots x_N$ where $x_1, x_2, \ldots, x_N$ are linearly independent random variables, then [7, p. 56]

$$E(y) = \prod_{i=1}^{N} E(x_i) \tag{7}$$

and

$$\sigma_y^2 = \prod_{i=1}^{N} [\sigma_{x_i}^2 + E(x_i)^2] - [\prod_{i=1}^{N} E(x_i)]^2 \quad . \tag{8}$$

Division of random variables is much more complicated. For example, if $y = 1/x$, the moments of the distribution of y are not definable in terms of the moments of the distribution of x [8, pp. 128-129]. Thus, the 1/P terms in equations 3 and 4 are potentially troublesome.

The approach adopted in the following analysis is to look for sums and products of linearly independent random variables and avoid quotients of random variables. Not infrequently, we will assume random variables to be linearly independent when in fact a weak relationship actually exists. This approximation is necessary if the analysis is to be tractable.

We will first consider the network probability of success as defined by equation 1. This equation can be rewritten as

$$P = \sum_{j=1}^{J} \sum_{k=1}^{K} a_{jk} \, p_{jk} \tag{9}$$

where $a_{jk} = \lambda_{jk}/\Lambda$ is the proportion of class j demand originating at requesting library k. Assuming the probability of success to be independent of the request source, equations 7 and 8 can be utilized to determine $E(P)$ and σ_P^2, respectively.

Assuming that the k^{th} requesting library generates n_k requests over some period of interest and that these are categorized into J classes of approximately equal size, then

$$E(a_{jk}) = \frac{n_k}{JN} \tag{10}$$

$$\sigma^2_{a_{jk}} = \frac{\frac{n_k}{JN}\left[1 - \frac{n_k}{JN}\right]}{N} \tag{11}$$

where

$$N = \sum_{k=1}^{K} n_k \ . \tag{12}$$

To determine the expected value and variance of p_{jk}, we first note that (see Appendix)

$$p_{jk} = \sum_{\ell=1}^{L} p_{jk\ell} \tag{13}$$

where $p_{jk\ell}$ is determined from model parameter $p_{ij|\ell}$ and a knowledge of the request routing policy employed. To avoid having to prescribe a particular request routing policy, it will be assumed that $p_{ij|\ell}$ does not vary with i. In other words, it will be assumed that all resource libraries are identical. This assumption results in

$$p_{jk\ell} = p_{j|\ell} \prod_{m=1}^{\ell-1} (1-p_{j|m}) \tag{14}$$

where $p_{j|\ell}$ is $p_{ij|\ell}$ with the i subscript dropped to reflect our assumption.

Equations 5 through 8 can then be utilized to compute $E(p_{jk})$ and $\sigma^2_{p_{jk}}$ in terms of $E(p_{j|\ell})$ and $\sigma^2_{p_{j|\ell}}$ which are given by

$$E(p_{j|\ell}) = p_{j|\ell} \tag{15}$$

$$\sigma^2_{p_{j|\ell}} = \frac{p_{j|\ell}(1-p_{j|\ell})}{n_{j|\ell}} \tag{16}$$

where $n_{j|\ell}$ is the sample size upon which the estimate of $p_{j|\ell}$ is based. If one assumes that requests are uniformly distributed among resource libraries at each stage of referral, then it is straightforward to show that

$$n_{j|\ell} = \frac{N}{IJ} \prod_{m=1}^{\ell-1} (1-p_{j|m}) \quad . \tag{17}$$

It would simplify matters substantially if one could choose values of $p_{j|\ell}$. In many situations, $p = 1/2$ maximizes uncertainty and thus, we will utilize $p_{j|\ell} = 1/2$ for all j and ℓ. With this assumption and recognizing two truncated geometric series [6, p. 99] that appear upon combining equations, one obtains

$$E(p_{jk}) = 1 - (1/2)^L \tag{18}$$

$$\sigma^2_{p_{jk}} = \sum_{\ell=1}^{L} (1/2)^{2\ell} \left[\prod_{m=1}^{\ell} (\frac{IJ}{N} 2^{m-1} + 1) - 1 \right] . \tag{19}$$

Thus, equations 10, 11, 18 and 19 are sufficient for calculation of σ^2_p. At this point, it is interesting to note that data collection involves two basic decisions: choosing J and N. In other words, one has to decide how much data to collect and how many request classes to utilize. Studying equations 10, 11, 18, and 19, one notes that increasing N always helps decrease σ^2_p. However, the effect of J is not as straightforward as $\sigma^2_{a_{jk}}$ and $\sigma^2_{p_{jk}}$ are effected in opposite ways by changes in J. We will return to this point in the discussion of an example.

Now, we will consider W, the average time from initiation of a request until the desired item is received by the requesting library, as defined by equation 2. To avoid the difficulties posed by the 1/P term, we will rewrite this equation as

$$PW = \sum_{j=1}^{J} \sum_{k=1}^{K} a_{jk}\, p_{jk}\, w_{jk} \tag{20}$$

where a_{jk} retains its previous definition. Our approach here will be to determine E(PW) and σ^2_{PW} and then utilize equations 7 and 8 to work backwards to obtain E(W) and σ^2_W. This is possible since P and W are linearly independent. (W is a function of P (equation 2), but the relationship is not linear.)

Utilizing equation A10, we obtain

$$PW = \sum_{j=1}^{J} \sum_{k=1}^{K} a_{jk}\, b_{jk} \tag{21}$$

where

$$b_{jk} = \sum_{\ell=1}^{L} p_{jk\ell}\, w_{jk\ell}\ . \tag{22}$$

From the Appendix, we note that $w_{jk\ell}$ is defined in terms of $\hat{w}_{ij\ell}$, $\tilde{w}_{ij\ell}$, and t_{ijk}. If we assume the expected values and variances of these model parameters to be $E(\hat{w})$, $E(\tilde{w})$, $E(t)$, $\sigma^2_{\hat{w}}$, $\sigma^2_{\tilde{w}}$, and σ^2_t respectively, then one obtains

$$E(w_{jk\ell}) = (\ell-1)E(\tilde{w}) + E(\hat{w}) + E(t) \tag{23}$$

$$\sigma^2_{w_{jk\ell}} = IJ\left[\frac{2(2^{\ell}-1)}{N}\sigma^2_w + \frac{1}{n_k(1-(1/2)^L)}\sigma^2_t\right] \tag{24}$$

where, as in the derivation of equations 18 and 19, the recognition of two truncated geometric series considerably simplifies the resulting expressions.

Knowing the moments of $w_{jk\ell}$ and the previously defined moments of $p_{jk\ell}$ and a_{jk}, one can use the properties of sums and products of linearly independent random variables, to determine $E(W)$ and σ^2_W. Since it will later be of use, we should also note that quite similar manipulations will yield $E(w_{jk})$ and $\sigma^2_{w_{jk}}$.

Now, we will consider C, the average cost per satisfied request, as defined by equation 3. In a manner similar to that with which we approached W, this equation will be rewritten as

$$PC = \sum_{j=1}^{J}\sum_{k=1}^{K} a_{jk}\, p_{jk}\, c_{jk} \quad . \tag{25}$$

As before, our approach is to determine $E(PC)$ and σ^2_{PC} and then work backwards to obtain $E(C)$ and σ^2_C.

Utilizing equations in the Appendix, we will define

$$d_{jk} = p_{jk}\, c_{jk} = p_{jk}L\tilde{c} + p^2_{jk}[\hat{c} - (L+1)\tilde{c}] + p_{jk}\, e_{jk} \tag{26}$$

where

$$e_{jk} = \sum_{\ell=1}^{L} p_{jk\ell}\, \ell\tilde{c} \tag{27}$$

with $\hat{c}$ and $\tilde{c}$ being the cost of satisfying and not satisfying, respectively, a request at any particular resource library in the network.

The moments of e_{jk} can be determined using equations 5 and 6 while the moments of d_{jk} are somewhat more complicated requiring fourth moments [8, p. 162] of what will be assumed to be a normal sampling distribution. Combining the moments of d_{jk}, e_{jk}, and our previous results, E(C) and σ_C^2 can be determined. Also of use are the moments of c_{jk} which are straightforward and given by

$$E(c_{jk}) = L\tilde{c} + [\hat{c} - (L+1)\tilde{c}]\, E(p_{jk}) + E(e_{jk}) \tag{28}$$

$$\sigma^2_{c_{jk}} = [\hat{c} - (L+1)\tilde{c}]^2\, \sigma^2_{p_{jk}} + \sigma^2_{e_{jk}} \tag{29}$$

We now can compute σ_P^2, σ_W^2, and σ_C^2 as a function of the uncertainty in our model parameters. Assuming that the sampling distributions of P, W, and C are normal, we can determine the 95% confidence intervals for each performance measure using $\pm$ 1.96 σ. In other words, we can determine an interval such that we are 95% confident that the true average performance measure lies in that interval.

To further refine our measure of uncertainty, we will define a modified coefficient of variation as 1.96 $\sigma/E(\cdot)$. Suppose this coefficient has some value x. Then, we could say that we are 95% confident that the true average performance is within 100 x percent of the estimated average performance.

AN EXAMPLE

In reference 2, we discuss an analysis of ILLINET based on a sample collected in February 1975. The sampling scheme resulted in accumulation of one day's demand from each of 19 requesting libraries. In terms of one year (250 working days), this was a 0.4% sample. This sample gives us estimates of n_k, $k = 1,2,\ldots,19$ for use in the procedure presented in this paper. The other input necessary to the procedure includes $E(\hat{w})$, $E(\tilde{w})$, $E(t)$, σ_w^2, σ_t^2, $\hat{c}$, and $\tilde{c}$. For these variables, we will choose realistic but arbitrary values of 5,5,5,5,5,1, and 1.

The modified coefficients of variation for P, W, and C are shown in Table I as a function of sample size (N) and number of request classes (J). Note that uncertainty is fairly sensitive to N but almost insensitive to J. We discussed the reason for this result in our derivation of σ_P^2. Even from a purely intuitive point of view, this result is not surprising since it would seem that our aggregate performance measures should not be too sensitive to disaggregation of component measures.

Using Table I, we can reach conclusions such as: At least a 2% sample (5 days) is necessary to maintain uncertainty under 10% for all performance measures. A law of diminishing returns is evident even in the rounded-off entries of this Table. For example, a 4% sample does not yield one half the uncertainty of a 2% sample.

To consider uncertainty on a less aggregated level, the modified coefficients of variation for p_{jk}, w_{jk}, and c_{jk} are shown in Table II.* Recall that j refers to request class while k refers to requesting library. We will call performance at this level type jk performance. As one would probably expect, uncertainty in type jk performance measures is quite sensitive to J.

Considering the results in Tables I and II, one might want to choose different accuracy goals at each level. For example, 5% uncertainty

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*The coefficient for w_{jk} was calculated using the average n_k.

SAMPLE SIZE (%YEAR)		NUMBER OF REQUEST CLASSES 1	2	3	4	5	6	7	8	9	10
0.4	P	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.08	0.08
	W	0.02	0.02	0.02	0.03	0.03	0.03	0.03	0.03	0.03	0.03
	C	0.20	0.20	0.21	0.21	0.21	0.22	0.22	0.22	0.23	0.23
0.8	P	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
	W	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02
	C	0.14	0.14	0.14	0.14	0.14	0.15	0.15	0.15	0.15	0.15
1.2	P	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04
	W	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
	C	0.11	0.11	0.12	0.12	0.12	0.12	0.12	0.12	0.12	0.12
1.6	P	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04
	W	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
	C	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
2.0	P	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03
	W	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
	C	0.09	0.09	0.09	0.09	0.09	0.09	0.09	0.09	0.09	0.09
2.4	P	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03
	W	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
	C	0.08	0.08	0.08	0.08	0.08	0.08	0.08	0.08	0.08	0.08
2.8	P	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03
	W	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
	C	0.07	0.07	0.07	0.07	0.07	0.08	0.08	0.08	0.08	0.08
3.2	P	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03
	W	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
	C	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07
3.6	P	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02
	W	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
	C	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07
4.0	P	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02
	W	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
	C	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06

TABLE I: Modified Coefficients of Variation for Network Performance

SAMPLE SIZE (%YEAR)		NUMBER OF REQUEST CLASSES									
		1	2	3	4	5	6	7	8	9	10
0.4	PJK	0.11	0.16	0.19	0.22	0.25	0.28	0.30	0.32	0.34	0.36
	WJK	0.09	0.13	0.16	0.18	0.21	0.23	0.25	0.27	0.29	0.30
	CJK	0.25	0.36	0.44	0.51	0.57	0.63	0.68	0.73	0.78	0.83
0.8	PJK	0.08	0.11	0.14	0.16	0.18	0.19	0.21	0.22	0.24	0.25
	WJK	0.06	0.09	0.11	0.13	0.14	0.16	0.17	0.18	0.20	0.21
	CJK	0.18	0.25	0.31	0.36	0.40	0.44	0.48	0.51	0.54	0.57
1.2	PJK	0.06	0.09	0.11	0.13	0.14	0.16	0.17	0.18	0.19	0.20
	WJK	0.05	0.07	0.09	0.10	0.12	0.13	0.14	0.15	0.16	0.17
	CJK	0.15	0.21	0.25	0.29	0.33	0.36	0.39	0.42	0.44	0.47
1.6	PJK	0.06	0.08	0.10	0.11	0.12	0.14	0.15	0.16	0.17	0.18
	WJK	0.04	0.06	0.08	0.09	0.10	0.11	0.12	0.13	0.14	0.14
	CJK	0.13	0.18	0.22	0.25	0.28	0.31	0.34	0.36	0.38	0.40
2.0	PJK	0.05	0.07	0.09	0.10	0.11	0.12	0.13	0.14	0.15	0.16
	WJK	0.04	0.06	0.07	0.08	0.09	0.10	0.11	0.11	0.12	0.13
	CJK	0.11	0.16	0.20	0.23	0.25	0.28	0.30	0.32	0.34	0.36
2.4	PJK	0.05	0.06	0.08	0.09	0.10	0.11	0.12	0.13	0.14	0.14
	WJK	0.04	0.05	0.06	0.07	0.08	0.09	0.10	0.10	0.11	0.12
	CJK	0.10	0.15	0.18	0.21	0.23	0.25	0.27	0.29	0.31	0.33
2.8	PJK	0.04	0.06	0.07	0.08	0.09	0.10	0.11	0.12	0.13	0.13
	WJK	0.03	0.05	0.06	0.07	0.08	0.08	0.09	0.10	0.10	0.11
	CJK	0.10	0.14	0.17	0.19	0.21	0.23	0.25	0.27	0.29	0.30
3.2	PJK	0.04	0.06	0.07	0.08	0.09	0.10	0.10	0.11	0.12	0.12
	WJK	0.03	0.04	0.06	0.06	0.07	0.08	0.08	0.09	0.10	0.10
	CJK	0.09	0.13	0.16	0.18	0.20	0.22	0.24	0.25	0.27	0.28
3.6	PJK	0.04	0.05	0.06	0.07	0.08	0.09	0.10	0.10	0.11	0.12
	WJK	0.03	0.04	0.05	0.06	0.07	0.07	0.08	0.09	0.09	0.10
	CJK	0.08	0.12	0.15	0.17	0.19	0.21	0.22	0.24	0.25	0.27
4.0	PJK	0.04	0.05	0.06	0.07	0.08	0.09	0.09	0.10	0.11	0.11
	WJK	0.03	0.04	0.05	0.06	0.06	0.07	0.08	0.08	0.09	0.09
	CJK	0.08	0.11	0.14	0.16	0.18	0.20	0.21	0.23	0.24	0.25

TABLE II: Modified Coefficients of Variation for Type jk Performance in network performance might be used with 10% uncertainty in type jk performance. One might then choose N so as to achieve the 5% goal for network performance. If the 10% goal for type jk performance is not achieved for the corresponding value of N, then J might be decreased. Of course, this approach is only one of a variety of possible methods for choosing N and J.

CONCLUSIONS

The problem considered in this paper is fairly complex, but nevertheless very important. In an attempt to deal with the complexity, we have resorted to various approximations. The result is what we feel to be a reasonable method for deciding how much data to collect and how the data should be aggregated.

The approach developed here is currently being used as a means of deciding when to terminate data collection in the study of a regional network within Illinois [4]. Further, it is being used to decide upon the number of request classes to use in the policy analyses being performed with our library network model.

Another interesting potential application of the method proposed here is in the study of time variations in average network performance. The sample size considerations resulting from analyses such as developed here put constraints on the type of time variations which can be studied. For example, if your accuracy requirements dictate at 10% sample, then one cannot study weekly variations of average demand unless, of course, one is willing to space his sampling out over the whole year.

Another data collection issue concerns how long one should sample to obtain a certain sample size. For example, one may have to collect data for 15% of the year to actually accumulate complete data on any particular period equal to 10% of the year. This is due to the fact that requests stay in a network for a period of time. A simple model which is of use in answering this question is discussed in reference 4.

The accuracy of the output of mathematical models is, for the most part, dictated by the accuracy of the input data or parameters. The purpose of this paper has been to suggest a method of dealing with this problem.

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APPENDIX

Since the publication of the paper in which our library network model was derived, we have developed a new version of the model that requires fewer assumptions and allows more robust representations of library networks. In this appendix, we will present the derivation of this new version of the model.

Assume that we have K requesting libraries who demand resources in J request classes. We will define λ_{jk}; $j = 1,2,\ldots,J$; $k = 1,2,\ldots,K$; as the average demand (per unit time) in class j from source k. This will be termed type jk demand.

Type jk demand will be routed through the network using a routing vector with elements $r_{jk\ell}$; $\ell = 1,2,\ldots,L_{jk}$; where the ℓ^{th} element of the routing vector is the designation of the resource library to which the request will be routed if it has not been satisfied at the ℓ-1 resource libraries specified by the first ℓ-1 elements of the routing vector.

Given the demand data and routing vectors (or policies), we would like to determine $\lambda_{ijk\ell}$ which is the average type jk demand (per unit time) on resource library i; $i = 1,2,\ldots,I$; at routing stage ℓ. This will be termed type ijkℓ demand.

To compute $\lambda_{ijk\ell}$, we must determine the probability that a type jk request will be satisfied at its ℓ^{th} routing stage. (This probability will be denoted by $p_{jk\ell}$.) For this to occur, the request must pass unsatisfied through its previous ℓ-1 routing stages. Thus,*

$$p_{jk\ell} = p(r_{jk\ell}, j|\ell) \prod_{m=1}^{\ell-1} [1-p(r_{jkm}, j|m)] \tag{A1}$$

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*Subscripts are sometimes bracketed for clarity. However, the meaning does not change. For example, $p_{ij|\ell}$ and $p(i,j|\ell)$ are equivalent. Also, note that for ℓ=1, $\prod_{m=1}^{\ell-1} (\ .\ ) \stackrel{\Delta}{=} 1.0$ and $\sum_{m=1}^{\ell-1} (\ .\ ) \stackrel{\Delta}{=} 0.0$.

where $p(i,j|\ell)$ is the probability that resource library i will satisfy a type $j\ell$ request. Note that we have assumed $p(i,j|\ell)$ to depend on the number of resource libraries that have processed the request previously. This reflects the possibility of requests with large ℓ being "difficult" in the sense that ℓ-1 resource libraries have been unable to satisfy them. While we might allow $p(i,j|\ell)$ to depend on the specific ℓ-1 resource libraries who have unsuccessfully processed the request, this would pose enormous data collection problems in that parameters for numerous unlikely routing vectors would have to be evaluated.

Resource library i can be described as a subnetwork of M_i subnodes with transitions between the subnodes described by transition probability matrix $\underline{P}_{ij\ell}$. Defining $\underline{\lambda}_{ij\ell}$ as an M_i vector which represents the average network demand directly entering the M_i subnodes,* then

$$\underline{\Lambda}_{ij\ell} = (\underline{I} - \underline{P}'_{ij\ell})^{-1} \underline{\lambda}_{ij\ell} \tag{A2}$$

where $\underline{I}$ is an $M_i \times M_i$ identity matrix, $(\cdot)'$ denotes the transpose of a matrix, $(\cdot)^{-1}$ denotes the inverse of a matrix, and $\underline{\Lambda}_{ij\ell}$ is an M_i vector whose m^{th} element represents the average type $j\ell$ demand on the m^{th} subnode of resource library i.

If we assume† that $\underline{P}_{ij\ell}$ is independent of $\underline{\lambda}_{ij\ell}$, then the fraction of $\underline{\lambda}_{ij\ell}$ reaching each subnode of resource library i is independent of

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*Since subnetworks can be arbitrarily defined, we can with no loss in generality assume that the first element of $\underline{\lambda}_{ij\ell}$ is its only non-zero element. This greatly aids in our computation of $p(i,j|\ell)$.

†If $\underline{P}_{ij\ell}$ is related to $\underline{\lambda}_{ij\ell}$, then these matrices will be interrelated among libraries and lead to our needing a simultaneous solution to a large number of nonlinear matrix equations.

$\underline{\lambda}_{ij\ell}$. Thus, we can compute $\underline{\Lambda}_{ij\ell}$ for an arbitrary $\underline{\lambda}_{ij\ell}$ and the fraction of $\underline{\lambda}_{ij\ell}$ reaching the subnode which represents request satisfaction is $p(i,j|\ell)$.

Given $p(i,j|\ell)$, we can use equation A1 to compute $p_{jk\ell}$ and then compute $\lambda_{ijk\ell}$ as follows. Let $\lambda_{ijk\ell} = 0$ for all i,j,k, and ℓ. Then, recursively use

$$\lambda(r_{jk\ell},j,k,\ell) = \lambda(r_{jk\ell},j,k,\ell) + \prod_{m=1}^{\ell-1} (1-p_{jkm})\lambda_{jk} \qquad \text{(A3)}$$

for all j,k, and ℓ. This now allows us to compute

$$\lambda_{ij\ell} = \sum_{k=1}^{K} \lambda_{ijk\ell} \qquad \text{(A4)}$$

and thereby define the first and only non-zero element of $\underline{\lambda}_{ij\ell}$. Equation A2 can then be employed to compute $\underline{\Lambda}_{ij\ell}$.

To estimate the average time required for a request to pass through resource library i, we first must estimate the average time required for a request to pass through each subnode of resource library i. From a queueing perspective, this requires an estimate of the total average demand on each subnode which is given by*

$$\underline{\Lambda}_i = \sum_{j=1}^{J} \sum_{\ell=1}^{L} \underline{\Lambda}_{ij\ell} \quad . \qquad \text{(A5)}$$

Defining $\underline{w}_i$ as an M_i vector representing the average times to pass through the subnodes of resource library i, then

$$\underline{w}_i = \underline{F}\ (\underline{\Lambda}_i, \text{ other parameters}) \qquad \text{(A6)}$$

---

*$L = \text{Max}\ \{L_{jk}\}$ or, more simply, L=I.

where $\underline{F}(\cdot)$ is a matrix function reflecting the particular queueing model chosen to represent subnode processing. Defining $\underline{L}_i$ as an M_i vector whose m^{th} element is the product of the m^{th} elements of $\underline{\Lambda}_{ij\ell}$ and $\underline{w}_i$, then

$$\underline{L}_{ij\ell} = (\underline{I} - \underline{P}'_{ij\ell})^{-1} \underline{L}_i \tag{A7}$$

where $\underline{L}_{ij\ell}$ is an M_i vector whose m^{th} element is the product of the m^{th} elements of $\underline{\Lambda}_{ij\ell}$ and $\underline{W}_{ij\ell}$ where $\underline{W}_{ij\ell}$ is an M_i vector representing the cumulative average time for a type $j\ell$ request to pass through each of the subnodes of resource library i. Since $\underline{\Lambda}_{ij\ell}$ is known, we can solve for $\underline{W}_{ij\ell}$.

Without loss of generality, we can assume that a single subnode of resource library i represents the departure point for satisfied requests while another single subnode represents the departure point for unsatisfied requests. Now, define $\hat{w}_{ij\ell}$ as the average time to successfully pass through the success subnode while $\tilde{w}_{ij\ell}$ is the average time to pass through that subnode which is the departure point for unsatisfied requests. These two variables are distinct elements of $\underline{W}_{ij\ell}$.

Now, let $w_{jk\ell}$ be the cumulative average time from initiation of a type jk request through its satisfaction at the ℓ^{th} stage of referral and the delivery of the desired document or information to the requesting library. This variable is calculated using

$$w_{jk\ell} = \sum_{m=1}^{\ell-1} \tilde{w}(r_{jkm}, j, m) + \hat{w}(r_{jk\ell}, j, \ell) + t(r_{jk\ell}, j, k) \tag{A8}$$

where t_{ijk} is the average time for a class j item to be delivered from resource library i to requesting library k.

With one modification, we can calculate average processing costs in the same manner as average processing times by simply substituting costs for times in equations A7 and A8. Thus, using $\underline{c}_i$ in equation A7 will lead to computation of $\underline{C}_{ij\ell}$ whose elements ($\hat{c}_{ij\ell}$ and $\tilde{c}_{ij\ell}$) can be used in equation A8. We have to employ A8 twice: once to calculate the cumulative cost of success $\hat{c}_{jk\ell}$ and once to calculate the cumulative cost of failure $\tilde{c}_{jk\ell}$ (by using $\hat{c}_{ij\ell} = \tilde{c}_{ij\ell}$ and zero delivery cost). The cost of failure is only calculated for $\ell = L_{jk}$ and reflects the fact that the overall cost of operating the network includes not only the investment in satisfied requests but also the funds invested in requests that were not satisfied.

We can now calculate the performance experienced by a type jk request. The probability that a type jk request is satisfied is given by

$$p_{jk} = \sum_{\ell=1}^{L_{jk}} p_{jk\ell} \tag{A9}$$

while the average time required from initiation of the request until receipt of the desired item is

$$w_{jk} = \sum_{\ell=1}^{L_{jk}} p_{jk\ell}\, w_{jk\ell}/p_{jk} \tag{A10}$$

and the average cost of satisfying a type jk request is given by

$$c_{jk} = \sum_{\ell=1}^{L_{jk}} p_{jk\ell}\, \hat{c}_{jk\ell} + (1-p_{jk})\tilde{c}_{jk L_{jk}} \tag{A11}$$

The performance measures p_{jk}, w_{jk}, and c_{jk} can be summed across j and/or k (weighted by λ_{jk}) to yield additional aggregate performance measures.

This new version of the library network model has been programmed in approximately 175 FORTRAN statements and allows for arbitrary network and subnetwork configurations.